McGill University
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE MATHEMATICS
Paper BETA

16 May, 2014
13:00 { 17:00

INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module. If you answer more than 7 questions, then clearly identify which 7 questions should be graded.

(ii) Pay careful attention to the exposition. Make an e ort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

Algebra Module

[ALG. 1] Prove that for $n \geq 5$, the alternating group A_n is the only non-trivial normal subgroup of the symmetric group on n elements, S_n.

[ALG. 2] Let G be a group of order 385. Show that the 7-Sylow subgroup of G is contained in the centre of G and that the 11-Sylow subgroup is normal.

[ALG. 3] Let $\mathbb{F}$ be a field. Consider a category **C** whose objects are pairs (V, T) where V is an $\mathbb{F}$-vector space and $T : V \to V$ is an $\mathbb{F}$-linear transformation.

(a) Explain how to define morphisms in **C** such that **C** becomes a category equivalent to the category of left $\mathbb{F}[x]$-modules.

(b) Given a pair (V, T) what is its torsion submodule under the equivalence in part (b)? Consider the particular example where $V = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{F}\}$ and $T((a_1, a_2, a_3 \ldots)) = (a_2, a_3, a_4 \ldots)$.

(c) Show that any finitely supported vector, i.e. a vector such that $a_n = 0$ for $n \gg 0$ is torsion.

(d) Show that the torsion of V contains other vectors.

[ALG. 4] Let $\mathbb{F}$ be a field of characteristic different from 2, and let $\mathbb{F}[x]$ the ring of polynomials in one variable over $\mathbb{F}$. Let $f(x) \in \mathbb{F}[x]$ be a non-constant separable polynomial of degree n. Write $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$ in some splitting field of f. Define the discriminant of f to be

$$D(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2.$$

(a) Prove that the discriminant of f lies in $\mathbb{F}$ and that the Galois group of f, viewed as a subgroup of S_n via its action on the roots of f, is contained in A_n if and only if $D(f)$ is a square in $\mathbb{F}$.

(b) Using that the discriminant of a cubic $f(x) = x^3 + ax + b$ is given by $-4a^3 - 27b^2$, prove that if $\mathbb{F}$ is a finite field then for every $a, b \in \mathbb{F}$ such that $x^3 + ax + b$ is irreducible over $\mathbb{F}$, the quantity $-4a^3 - 27b^2$ is a square in $\mathbb{F}$.

Analysis Module

[AN. 1] Given the measure space (X, μ) with $\mu(X) = 1$, suppose that there are μ-measureable subsets $E_n \subset X$, $n = 1, 2, \ldots$, whose complements $X - E_n$ are disjoint. Prove that

$$\lim_{N \to \infty} \prod_{n=1}^{N} \mu(E_n) \geq \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

(Hint: First prove by induction that if the y_k are ≥ 0 and $y_1 + y_2 + \cdots + y_n \leq 1$, then $(1 - y_1)(1 - y_2)\cdots(1 - y_l) \geq 1 - y_1 - y_2 - \cdots - y_l$ for $1 \leq l \leq n$.)

[AN. 2] Show that

$$\lim_{n \to \infty} \int_0^n \Big(1 - \frac{x}{n}\Big)^n dx$$

exists and find that limit. (Hint: First verify, e.g., using differential calculus, that $1 - t \leq e^{-t}$ for $0 \leq t \leq 1$.)

[AN. 3] Let $p > 1$, $f(x) \in L_p(\mathbb{R})$, and $\{a_n\} \in l_q(\mathbb{Z})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Show that the series $\sum_{-\infty}^{\infty} a_n f(x - n)$ converges absolutely for almost all x to a sum $F(x)$ with $\int_0^1 |F(x)|^p dx < \infty$. Estimate the last integral in terms of $\sum_{-\infty}^{\infty} |a_n|^q$ and $\|f\|_p$.

[AN. 4]

(a) Compute
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|\xi|}\operatorname{sgn}\xi \, e^{i\xi x} d\xi;$$
where $y > 0$, $x \in \mathbb{R}$.

(b) Hence show that for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$,
$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x - t}{(x - t)^2 + y^2} f(t)dt = \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{-y|\xi|} \hat{f}(\xi) e^{i\xi x} dx$$
(when $y > 0$ and $x \in \mathbb{R}$).

(c) Conclude that if
$$\varphi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x - t)}{(x - t)^2 + y^2} f(t)dt;$$
we have $\|\varphi\|_2 \leq \|f\|_2$.

Geometry and Topology Module

[GT. 1]

Let X be the topological space whose points are $\{a, b, x, y\}$ and whose topology is:

$$\{\{a, b, x, y\}, \{a, x, b\}, \{a, y, b\}, \{a, b\}, \{a\}, \{b\}, \{\}\}$$

(a) Prove that X is path connected.
(b) What is $\pi_1(X; a)$?
(c) Describe the universal cover $\tilde{X}$ of X, by indicating a basis for the topology of $\tilde{X}$.
(d) Explain why X does not have the same homotopy type as the circle S^1.

[GT. 2]

Let M be a moebius strip with boundary, and let $p \in M - \partial M$ be a point in its interior. Let $N = M - \{p\}$.

(a) Draw a connected degree two covering space $\hat{N}_1$ of N with the property that $\partial \hat{N}_1$ consists of exactly one circle.
(b) Draw an orientable degree two covering space $\hat{N}_2$ of N.
(c) Find three embedded closed curves in N that are not homotopic to each other.
(d) Identify four distinct compact surfaces that are not homeomorphic to each other, but have the same homotopy type as N.

[GT. 3]

(a) Show that a manifold is orientable if and only if the complement of the zero-section in the top exterior power of its tangent bundle has two components.
(b) One can produce the real projective plane in two different ways: as a quotient of the standard two-sphere in $\mathbb{R}^3$ by the involution $p \to -p$, and by glueing a disk to the boundary of the Moebius strip. Is the projective plane orientable? Compute its fundamental group, and its deRham cohomology.

[GT. 4]

(a) Consider the three vector fields on $\mathbb{R}^3$:

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}; \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}; \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that they form a Lie algebra.
(b) This Lie algebra is the Lie algebra of $SO(3)$. As such, $X; Y; Z$ can be thought of as left-invariant vector fields on the group $SO(3)$, trivializing its tangent bundle. Let $\omega_X; \omega_Y; \omega_Z$ be the dual basis of left-invariant one forms: $\omega_X(X) = 1; \omega_X(Y) = 0;$ etc.. Compute $L_X(\omega_X); L_Y(\omega_X); L_Z(\omega_X)$.
(c) Using part (b) or otherwise, compute $d\omega_X$ as a linear combination of exterior products of $\omega_X; \omega_Y; \omega_Z$.