Analysis Module

[AN. 1]
Let $\varepsilon > 0$ be fixed. Show that the set of all real numbers $x \in [0; 1]$ such that there exist infinitely many pairs $p; q \in \mathbb{N}$ such that $|x - p/q| < 1/q^{2+\varepsilon}$ has Lebesgue measure 0.

[AN. 2]
Let f be a uniformly continuous function on $\mathbb{R}$. Suppose that $f \in L^p$ for some p, $1 \le p < \infty$. Prove that $f(x) \to 0$ as $|x| \to \infty$.

[AN. 3]
(a) Give a definition of $||f||_\infty$ of a measurable complex function f.

(b) Recall that the essential range of a function $f \in L^\infty(\mu; \mathbb{C})$ is the set consisting of complex numbers w such that

$$\mu(\{x : |f(x) - w| < \varepsilon\}) > 0$$

for every $\varepsilon > 0$. Prove that R_f is compact.

(c) Show that $||f||_\infty = \sup_{w \in R_f} |w|$.

[AN. 4]
(a) Give a definition of a locally compact topological space.

(b) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}; \mu)$ is locally compact and explain why it is so.

(c) Give an example of a Borel measure μ on $\mathbb{R}$ such that $X = L^2(\mathbb{R}; \mu)$ is not locally compact and explain why it is so.

Numerical Analysis module

[NA. 1] *Quadrature and Newton's Method*

Let $f(x) = \frac{1}{4}(x - 5)^4 + x$.

(a) Compute $f'(x), f''(x)$. Is f convex? Explain your answer.

(b) Find the minimizer of $f(x)$.

(c) Write out the formula for Newton's method for function minimization.

(d) Compute two Newton iterations, for $x^0 = 4.5$. Are the values approaching the minimum?

(e) Approximate the integral $\int_0^3 \frac{1}{x^2+2}\, dx$

[NA. 3] The conserved quantity q with ux function F satis es the conservation law

$$\frac{\partial}{\partial t} q(x,t) + \frac{d}{dx} F(q,x,t) = 0, \qquad \text{for } x \in [0,1] \tag{1}$$

along with no- ux boundary conditions

$$F(q,x,t) = 0, \qquad \text{for } x = 0,1.$$

(a) Show that the total mass of q is conserved.

(b) Assume that Fick's law of di usion holds, so that $F(q(\ ),x,t) = \ \ (x) q_x(x,t)$. The energy is $E(t) = \frac{1}{2}\int_0^1 q^2(x,t)dx$. Prove that the energy is non-increasing.

(c) Let $G = [0,h,\dots,1]$ be the nite di erence grid, where $h = 1/(n \ \ 1)$. Let ∂_x^h be the forward di erence operator on the grid. Let $Q = (Q_0,\dots,Q_n)$ be a grid function. Write down the matrix M which maps the grid function Q to the grid function $\partial_x^h Q$, and includes the boundary conditions.

(d) Let $Q(t) = (Q_0(t),\dots,Q_n(t))$ be a time-dependent grid function. Consider the method of lines for the PDE,

$$\frac{d}{dt} Q + M^{|} (diag(\ ) M Q)$$

Prove that mass is conserved, and that the discrete energy $E^h(t) = \frac{h}{2}\langle Q(\ ,t), Q(\ ,t)\rangle$ is non-increasing.

[NA. 4]

(a) Consider the initial value problem for the variable coe cient parabolic equation on the real line

$u_t(x,t) + f(x,t)u_x(x,t) =$ fan tmark1 variable-19.98 9. 9.9626 Tf 242.264 0 Td [(F)rabolic equa

Partial Differential Equations Module

[PDE 1.] We consider the boundary value problem

$$\begin{cases} \partial_y u + (2x + u)\,\partial_x u = x + 2u & \text{in } U \\ u(x, \alpha x) = 1 + \beta x & \text{on } \partial U \end{cases} \qquad (P)$$

where $U = \{(x, y) : y > \alpha x\}$ and $\alpha, \beta \in \mathbb{R}$

(a) For which values of α and β does the problem (P) satisfy the noncharacteristic boundary condition?
(b) Give all solutions of the problem (P) in case $\alpha = 0$ and $\beta = -1$.
(c) Show that there does not exist any solution of the problem (P) in case $\alpha = 1$ and $\beta \neq 2$.

[PDE 2.]

(a) Let U be an open and bounded subset of $\mathbb{R}^n$, $n \geq 1$. Show that for any functions $u, v \in C^2(U) \cap C^0(\overline{U})$ such that $-\Delta u \leq -\Delta v$ in U and $u \leq v$ on ∂U, we have $u \leq v$ in U.
(b) Now we assume that $n = 2$ and $U = \{x \in \mathbb{R}^2 : R_1 < |x| < R_2\}$ for some real numbers $R_2 > R_1 > 0$. Show that for any function $u \in C^2(U) \cap C^0(\overline{U})$ such that $-\Delta u \leq 0$ in U, we have

$$M(r) \leq \frac{M(R_1)\ln(R_2/r) + M(R_2)\ln(r/R_1)}{\ln(R_2/R_1)} \quad \forall r \in (R_1, R_2)$$

where $M(r) = \sup\{u(x) : |x| = r\}$.
Hint: Remember that the function $v(x) = a + b\ln|x|$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ for all $a, b \in \mathbb{R}$.

[PDE 3.] Let U be an open and bounded subset of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary. We consider the problem

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } U \times (0, \infty) \end{cases}$$